Name: Feedback

SA402 · Dynamic and Stochastic Models — Fall 2023 · Uhan

# Exam 3 – 12/1/2023

## Instructions

- You have 50 minutes to complete this exam.
- You may use your plebe-issue TI-36X Pro calculator.
- You may use the provided list of formulas.
- You may not use any other materials.
- **No collaboration allowed.** All work must be your own.
- **Show all your work.** To receive full credit, your solutions must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until it is returned to you.**

| Problem | Weight | Score |
|---|---|---|
| 1a | 1 | |
| 1b | 1 | |
| 2a | 2 | |
| 2b | 1 | |
| 2c | 1 | |
| 2d | 1 | |
| 3 | 1 | |
| 4a | 1 | |
| 4b | 1 | |
| Total | | / 100 |

**Problem 0.** Copy and sign the honor statement below. This exam will not be graded without a signed honor statement.

The Naval Service I am a part of is bound by honor and integrity. I will not compromise our values by giving or receiving unauthorized help on this exam.

Signature:

**Problem 1.** The Simplexville branch of Bernoulli Bank has 2 tellers on duty. Customers wait in a single queue and are served by the first available teller, first-come first-served.

Potential customers arrive at the door of the bank a rate of 30 per hour. The interarrival times are exponentially distributed. When the bank has 10 or more customers total, potential customers balk with probability 1/3. The bank can only hold 20 customers total, including the 2 customers being served. Any potential customers that arrive when the bank is full simply go elsewhere.

The average service time for a single teller is 6 minutes per customer. The service times are exponentially distributed. Customers in the queue sometimes renege: the time a customer is willing to spend in the queue is exponentially distributed with a mean of 10 minutes.

Model this setting as a birth-death process by answering the following prompts.

a. Define the arrival rate in each state, in terms of the number of customers per hour.

Note that there are 3 different possibilities for arrivals in this scenario:

- When there are 0-9 customers in the bank, all potential customers enter the bank.
- When there are 10-19 customers in the bank, 1/3 of the potential customers balk.
- When there are 20 or more customers in the bank, the bank is full, so none of the potential customers enter the bank.

See page 4 of Lesson 14 for examples of how to model balking customers and systems with limited capacity.

b. Define the service rate in each state, in terms of the number of customers per hour.

Note that there are 2 different possibilities for departures in this scenario:

- When there are 1 or 2 customers in the bank, each customer is served by a teller at a rate of 10 customers/hour.
- When there are 3 or more customers in the bank, then
  - there are 2 customers being served by a teller; each of these customers departs at a rate of 10 customers/hour.
  - the remaining customers are in the queue and can renege; each of these customers departs at a rate of 6 customers/hour.

See page 5 of Lesson 14 for an example of how to model reneging customers with multiple identical servers.

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**Problem 2.** Dantzig's Donut Shop has a single queue for waiting customers and 2 cashiers. One of the cashiers is on duty at all times. The other cashier goes on duty whenever the shop becomes busy; in particular, when the shop has 3 or more customers (including the customers being served). The shop can hold at most 6 customers total.

Suppose that customer arrivals are well modeled as a Poisson process with a rate of 10 customers per hour. The cashiers both work at a rate of 6 customers per hour, and the service times are well modeled as exponential random variables.

This setting can be modeled as a birth-death process with the following arrival and service rates (in customers per hour):

$$\lambda_i = \begin{cases} 10 & \text{for } i = 0, 1, \dots, 5 \\ 0 & \text{for } i = 6, 7, \dots \end{cases} \qquad \mu_i = \begin{cases} 6 & \text{for } i = 1, 2 \\ 12 & \text{for } i = 3, 4, \dots \end{cases}$$

a. Over the long run, what is the probability that there are $n$ customers in the system ($n = 0, 1, \dots, 6$)? Provide your answers to 3 decimal places.

See Example 1 in Lesson 15 for a similar example. Some things to be careful about:

- In this scenario, the shop can hold at most 6 customers total, so $\pi_0, \pi_1, \dots, \pi_6$ should all be non-zero.
- Remember to include $d_0 = 1$ when computing $D$.

For the remaining parts of this problem, assume the steady-state probabilities are as given below:

$$\pi_0 = 0.08 \qquad \pi_1 = 0.13 \qquad \pi_2 = 0.22 \qquad \pi_3 = 0.18$$
$$\pi_4 = 0.15 \qquad \pi_5 = 0.13 \qquad \pi_6 = 0.11$$

These values may or may not match what you found in part a.

b. Over the long run, what fraction of time are both cashiers busy?

See Example 1 in Lesson 15 for a similar example. Note that in this scenario, however, that both cashiers are busy when there are 3 or more customers.

c. Over the long run, what is the expected number of customers in the shop?

See Problem 2c from the Review Problems for Exam 3 for a similar example.

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d. Over the long run, what is the expected time a customer spends in the shop? Provide your answer to 3 decimal places.

See Problem 2d from the Review Problems for Exam 3 for a similar example.

**Problem 3.** Sean and Dan's Pizza has 3 cashiers at its Simplexville location. Customers wait in a single queue and are served by the first available cashier, first-come first-served. The average service time is 2 minutes per customer, and customers arrive at a rate of 24 per hour. The interarrival times are best modeled with the exponential distribution. On the other hand, the service time distribution is unknown. The Simplexville location is enormous and popular, so for all intents and purposes, the restaurant has infinite capacity and has an infinite number of possible customers.

Which standard queueing model fits this setting best? No explanation is necessary.

See page 1 of Lesson 16 for a reminder of how standard queueing notation works.

Note that in this scenario, the service time distribution is unknown. Therefore, the best we can say here is that the service time follows a general distribution.

**Problem 4.** You have been asked to take over the task of staffing maintenance teams at a facility that repairs and maintains MH-53E Sea Dragon helicopters. The helicopters arrive at the maintenance facility at a rate of 4 per week, and are processed on a first-come-first-served basis. It takes a maintenance team 1 day on average to repair 1 helicopter. The facility currently has 1 maintenance team working at any given time.

Model this system as an M/M/1 queue. For the problems below, use weeks as your time unit.

a. Over the long run, what is the fraction of time is the maintenance team idle? Provide your answer to 3 decimal places.

See Example 4 in Lesson 16 for an example of computing $\pi_0$ for an M/M/1 queue.

b. Suppose you find that $\pi_0 = 0.429$. (This may or may not match what you found in part a.) What is the expected time in queue (i.e., expected delay) for a helicopter that comes in for repair? Provide your answer to 3 decimal places.

See Example 4 in Lesson 16 for an example of computing $\ell_q$ for an M/M/1 queue.

Note that for any M/M/$s$ queue with arrival rate $\lambda$, $\lambda_{\text{eff}} = \lambda$. Why? The arrival rate for this queueing system when there are $i$ customers in the system is:

$$\lambda_i = \lambda \qquad \text{for } i = 0, 1, 2, \dots$$

Therefore,

$$\lambda_{\text{eff}} = \sum_{i=0}^{\infty} \lambda_i \pi_i = \sum_{i=0}^{\infty} \lambda \pi_i = \lambda \underbrace{\sum_{i=0}^{\infty} \pi_i}_{=1} = \lambda$$